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DAMPING CONFIGURATIONS THAT HAVE A STABILIZING INFLUENCE ON NONCONSERVATIVE SYSTEMS

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Abstract—A particular damping configuration which always has a stabilizing influence on a nonconservative system is studied. It is seen that for very small damping the critical value of the chief parameter for incipient flutter tends to that of the undamped system.

In the simplest case of two degrees of freedom a range of stabilizing damping configurations is sought.

NOTATION

$\mathbf{a}$	inertia matrix
A	$(\Gamma_{22}-\Gamma_{11})_{P=P_c}$
$\mathbf{b}$, $\mathbf{b}(P)$	damping matrices
b_{ij}	coefficient of damping matrix
B	$(\Gamma_{12}+\Gamma_{21})_{P=P_c}$
$\mathbf{c}$, $\mathbf{c}(P)$	stiffness matrices
c_{ij}	coefficient of stiffness matrix
F_1, $F_2(\lambda, P)$	polynomials in λ
$\mathbf{I}$	unit matrix
$k(P)$	positive scalar
n	number of degrees of freedom
p_i	polynomial coefficient in characteristic equation
P	chief parameter
P_c	critical value of P for incipient flutter of the undamped system
$\mathbf{q}$	vector of generalized co-ordinates
$\mathbf{Q}(t)$	vector of generalized forces
t	time
T_{2n-1}	Routh's test function for incipient flutter
u, v, w	number of rows of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ involved in a determinantal expansion
V	airspeed
Γ_{ij}	coefficient in reduced stiffness matrix
δ	denotes small order
ε	positive scalar that can be made small
τ_{ij}	coefficient in reduced damping matrix
η_{ij}	deviation of τ_{ij} from that given by (41)
λ	complex variable
λ_i	ith eigenvalue
Λ	$\lambda/(c_{11}+c_{22})^{\frac{1}{2}}$
μ	$\tau_1{}^2/\tau_{11}\tau_{22}$ where $\tau_{12}=\tau_{21}$
σ_{ij}	that part of p_i dependent on damping coefficients to the jth power
$\mathbf{\Phi}$	component of $\mathbf{q}$ independent of t
$\mathbf{\Phi}_i$	ith eigenvector

1. INTRODUCTION

MANY stability studies of nonconservative systems have undoubtedly been initiated by the paper of Zeigler [1], in which the paradoxical destabilizing effect of damping is demonstrated. The type of system generally considered in these studies is a linear mechanical one subjected to nonconservative forces of the circulatory category (to use Zeigler's nomenclature [2]) which are monotonically dependent on one or more chief parameters. Depending on the problem a typical chief parameter could be either the magnitude of a follower force applied to the system structure, or the flow velocity in an elastic pipe conveying fluid, or the airspeed in an aerolastic problem. The fundamental property of the nonconservative system of this type is that at some critical value or values of the chief parameter or parameters the system becomes incipiently unstable, the instability being an oscillatory one and commonly referred to as flutter. As a result of the work by Zeigler [1] and other investigators (see the bibliography in Herrmann [3]) it is now well known that the lowest critical value of the chief parameter for flutter is generally reduced by the introduction of a small amount of damping into the otherwise undamped system. In this sense the added damping is destabilizing although the rate of growth of the divergent oscillations may be only of the same order of smallness of that of the damping [4].

The tendency in some of the literature to concentrate on the unusual and interesting destabilizing phenomenon e.g. [4–7] is such that it is easy to fall into the way of thinking that introducing small damping into a nonconservative system *always* brings about a loss of stability. This is not the case and examples may be found as testimony. Bolotin [8, p. 81] studied a two degree of freedom system in which the damping coefficients corresponding to appropriately transformed equations of motion could be made equal and demonstrated that in this case vanishingly small damping was not destabilizing. The same result applied for a system having equal eigenvalues; indeed, it was concluded for this that small but finite damping was strictly stabilizing.

As part of his investigation into the dynamics of tubular cantilevers Paidoussis [9] varied the structural and fluid generated damping; although some configurations of cantilever showed small damping to be destabilizing, others indicated the opposite effect. The dependence of the tendency to stabilize or not on the configuration of the basic structure was recognized in an early paper by Frazer [10] who studied the power input required to maintain forced oscillations of an elastic aeroplane wing in flight. He found conditions that applied to the system inertias such that the rate of change of the critical airspeed for flutter with increasing damping would be either positive or negative. Experiments were carried out with a small model wing in a wind-tunnel which provided limited but ample confirmation of the theory.

It is with the purpose of redressing the balance towards the point of view that the damping may be stabilizing in certain situations that this paper is written. Subsequent to the establishment of useful general relationships, a particular combination of damping coefficients embodied in matrix form (referred to here as a configuration of damping) that has the property of always being stabilizing is studied and in the following section a greater range of stabilizing configurations is sought.

2. GENERAL RELATIONSHIPS

The systems considered are assumed linear and scleronomic and the non-dissipative nonconservative forces are circulatory and dependent on one chief parameter only. The damping forces are velocity dependent.

The equations of motion in the undamped cases are

$$\mathbf{a}\ddot{\mathbf{q}} + \mathbf{c}(P)\mathbf{q} = \mathbf{Q}(t) \tag{1}$$

where **a** is an inertia matrix, **c** is a stiffness matrix in which the elements are functions of P, the chief parameter, **q** are generalized co-ordinates and $\mathbf{Q}(t)$ are the time dependent generalized forces. The solution $\mathbf{q} = \mathbf{\Phi}\, e^{\lambda t}$, where λ may be complex, applied to the homogeneous equations provides the characteristic equation in determinantal form

$$|\mathbf{a}\lambda^2 + \mathbf{c}(P)| = 0 \tag{2}$$

which may be expanded to give in polynomial form

$$F_1(\lambda^2, P) = p_{2n}\lambda^{2n} + p_{2n-2}\lambda^{2n-2} + \ldots + p_2\lambda^2 + p_0 = 0 \tag{3}$$

where n is the number of degrees of freedom present. The coefficients p_i are functions of P (except for p_{2n}) and λ appears only in even powers in the equation. The condition of incipient oscillatory instability is given by the coalescence of two conjugate pairs of λ-roots in the complex plane, which in turn derives from the coalescence of a real pair of λ^2-roots.

This happens when

$$\begin{aligned} F_2(\lambda^2, P) &= \partial F_1(\lambda^2, P)/\partial \lambda^2 \\ &= np_{2n}\lambda^{2n-2} + (n-1)p_{2n-2}\lambda^{2n-4} + \ldots 2p_4\lambda^2 + p_2 = 0 \end{aligned} \tag{4}$$

which is a particular case of the general statement by Bolotin and Zhinzher [12, p. 969].

We now consider the introduction of damping in the form $\varepsilon\mathbf{b}(P)\dot{\mathbf{q}}$ into the equations of motion; the elements of **b** may vary with P whilst the positive scalar ε can be made arbitrarily small. The determinantal form of the characteristic equation is

$$|\mathbf{a}\lambda^2 + \varepsilon\mathbf{b}(P)\lambda + \mathbf{c}(P)| = 0 \tag{5}$$

and the associated polynomial form is

$$p_{2n}^*\lambda^{2n} + p_{2n-1}^*\lambda^{2n-1} + p_{2n-2}^*\lambda^{2n-2} + \ldots + p_2^*\lambda^2 + p_1^*\lambda + p_0^* = 0 \tag{6}$$

where the asterisk indicates a change in a coefficient p_i at any given P due to the introduction of the damping. The new coefficients can be expressed as polynomials in ε, but to do this we must examine their formation from the elements of **a**, **b** and **c**. A particular p_i is given by the summation of determinants formed from, say, u rows of **a**, v rows of **b** and w rows of **c** in such a manner that the conditions

$$\left.\begin{aligned} u + v + w &= n \\ 2u + v &= i \end{aligned}\right\} \tag{7}$$

are satisfied. Clearly, for some values of i several combinations of the integers u, v and w are possible and all possible combinations are used in the summation.

The number of rows of **b** and hence the power to which the scalar appears in a combination is v which from (7) is

$$v = 2n - i - 2w. \tag{8}$$

The possible values of v for a given i are found by allowing w to vary, subject to the constraints $v \geq 0$, $w \geq 0$. We can deduce from (8) that v is even if i is even, and it is odd if i is odd, and that the maximum value for v is $2n - i$. Thus

$$p_i^* = p_i + \sigma_{i2}\varepsilon^2 + \ldots + \sigma_{i,2n-i}\varepsilon^{2n-i} (i \text{ even}) \tag{9}$$

$$p_i^* = \sigma_{i1}\varepsilon + \sigma_{i3}\varepsilon^3 + \ldots + \sigma_{i,2n-i}\varepsilon^{2n-i} (i \text{ odd}) \tag{10}$$

where the coefficients σ_{ij} are functions of P.

The addition of damping, whilst making the characteristic equation more complicated allows the condition for incipient oscillatory instability to be stated more simply than in the undamped case since the real parts of the exponents are negative in the stable range of P but at least one is positive in the unstable range. Impending flutter is indicated when the appropriate real part is zero i.e. λ is purely imaginary. This allows an easy division into real and imaginary parts of (6) so that for incipient flutter

$$F_1^*(\lambda^2, P) = p_{2n}^*\lambda^{2n} + p_{2n-2}^*\lambda^{2n-2} + \ldots p_2^*\lambda^2 + p_0^* = 0 \tag{11}$$

$$F_2^*(\lambda^2, P) = p_{2n-1}^*\lambda^{2n-2} + \ldots p_3^*\lambda^2 + p_1^* = 0 \tag{12}$$

where λ is taken to be non-zero. Equations (11) and (12) can be rearranged to eliminate λ^2 and all its powers to produce in the damped case the familiar Hurwitz determinantal form of the $(2n-1)$th Routh test function for oscillatory instability. The equivalent expansion of Collar (see [11]) for the undamped case is similarly formed from equations (3) and (4) to produce

$$T_{2n-1} = \begin{vmatrix} np_{2n} & (n-1)p_{2n-2} & (n-2)p_{2n-4} & \cdots & & & \\ p_{2n} & p_{2n-2} & p_{2n-4} & \cdots & & & \\ 0 & np_{2n} & (n-1)p_{2n-2} & \cdots & & & \\ 0 & p_{2n} & p_{2n-2} & \cdots & & & \\ \vdots & \vdots & \vdots & \vdots & \cdot & \cdot & \cdot \\ & & \cdots & 2p_4 & p_2 & 0 & \\ & & \cdots & p_4 & p_2 & p_0 & \\ & & \cdots & 3p_6 & 2p_4 & p_2 & \end{vmatrix} = 0. \tag{13}$$

3. NON-DESTABILIZING SMALL DAMPING CONFIGURATION

We now consider the damping to be very small so that the second and higher order terms in ε in (9) and (10) may be neglected. Then, for i even, $p_i^* \to p_i$ implying that $F_1^*(\lambda^2, P) \to F_1(\lambda^2, P)$ and for i odd, $p_i^* \to \varepsilon\sigma_{i1}$. For the critical value of the chief parameter to remain unchanged when damping is introduced into the initially undamped system the conditions represented by (3), (4), (11) and (12) must hold simultaneously; with small damping the reduced set of conditions is

$$F_1(\lambda^2, P) = F_2(\lambda^2, P) = F_2^*(\lambda^2, P) = 0. \tag{14}$$

Damping terms appear only in F_2^* and the problem is to find a configuration of $\mathbf{b}$ such that $F_2^*(\lambda^2, P) = 0$ is satisfied at the appropriate critical values of λ^2 and P. The damping matrix contains n^2 potentially variable parameters and thus in the absence of sufficient constraining relations, there is an infinity of possible configurations for $\mathbf{b}$ that satisfies (14).

One configuration that is acceptable is that for which the damping matrix is proportional to the inertia matrix i.e. $\mathbf{b} = k(P)\mathbf{a}$ where $k(P)$ is a positive scalar that may be dependent on P. This configuration encompasses that of Bolotin previously referred to and also was used by Done [13] as a means of acquiring insight into certain aircraft flutter problems. The determinantal form of the characteristic equation is in this case

$$|\mathbf{a}\lambda^2+k(P)\mathbf{a}\lambda+\mathbf{c}(P)| = 0. \tag{15}$$

The coefficients p_i^* of the polynomial form of the characteristic equation do not now contain any elements of **b**, and it is possible to express the coefficients in terms of those of the undamped system. Denoting, as before, the integers u, v and w for the number of rows of **a**, **b** and **c** involved in any determinantal expansion, subject to conditions (7), we observe that in the undamped system $v = 0$ so that $2u = i$ and each polynomial coefficient $p_i = p_{2u}$ is determined by summations having unique numbers of rows of **a** and **c**. In obtaining the polynomial form of (15) we replace v rows of **b** by v rows of **a** in each determinant formation so that associated with coefficients p_i^* in the damped system are coefficients p_{i+v} in the undamped case. However, the evaluation of a particular determinant may be repeated, as when the v rows of **b** are replaced by rows from **a**, the combination of rows of **a** and **c** may appear several times. The number of times is equal to the number of different orders that can be found for two sets of u and v rows i.e. $(u+v)!/u!v!$ Also, the power of ε and $k(P)$ is equal to the number of rows of **b** that were originally in a determinant formation. Thus, we can write

$$p_i^{**} = \sum_v \frac{(u+v)!}{u!v!}\varepsilon^v k^v(P) p_{u+v} \tag{16}$$

where p_i^{**} refers to the system with the damping matrix proportional to the inertia matrix and the summation is over all possible values of v subject to conditions (7) and u, v and w remaining non-negative. A coefficient of the type appearing in (9) and (10) becomes

$$\sigma_{iv}^{**} = \frac{(\frac{1}{2}i+\frac{1}{2}v)!}{(\frac{1}{2}i-\frac{1}{2}v)!v!}k(P)^v p_{i+v} \tag{17}$$

where i and v are either both odd or both even.

The case of damping made so small that terms involving second and higher order in ε in (9) and (10) may be neglected is given by making $v = 1$. Then from (10) and (17)

$$p_i^{**} = \varepsilon\sigma_{i1}^{**} = \tfrac{1}{2}(i+1)\varepsilon k(P) p_{i+1} \tag{18}$$

for odd i, and this allows (12) to be written

$$\begin{aligned} F_2^{**}(\lambda^2, P) &= p_{2n-1}^{**}\lambda^{2n-2}+\ldots p_3^{**}\lambda^2+p_1^{**} \\ &= \varepsilon k(P)\{np_{2n}\lambda^{2n-2}+(n-1)p_{2n-2}\lambda^{2n-4}+\ldots 2p_4\lambda^2+p_2\} = 0. \end{aligned} \tag{19}$$

Apart from the scalar $\varepsilon k(P)$ this function is the same as that for the undamped system $F_2(\lambda^2, P)$ in (4) and thus (14) is automatically satisfied at all times. Therefore, the introduction of a vanishingly small amount of damping in the form $\mathbf{b} = \varepsilon k(P)\mathbf{a}$ does not change the critical value of the chief parameter.

The corresponding test function for oscillatory instability can be shown to be

$$T_{2n-1}^{**} = \varepsilon^n k(P)^n T_{2n-1} = 0 \tag{20}$$

where T_{2n-1} is the test function given in (13) for the undamped system.

In order to examine the form of the eigenvalues (15) may be written for the ith eigenvalue

$$|\mathbf{a}\lambda_i^{**2} + k(P)\mathbf{a}\lambda_i^{**} + \mathbf{c}(P)| = 0 \tag{21}$$

in which the damping is not necessarily small and ε is omitted.

The ith eigenvalue in the undamped system satisfies

$$|\mathbf{a}\lambda_i^2 + \mathbf{c}(P)| = 0 \tag{22}$$

and satisfaction of both (21) and (22) at the same value of P is ensured if

$$\lambda_i^{**2} + k(P)\lambda_i^{**} = \lambda_i^2 \tag{23}$$

for each set of eigenvalues. This is a simple relationship which leads to

$$\lambda_i^{**} = -\frac{k(P)}{2} \pm \sqrt{\left[\left(\frac{k(P)}{2}\right)^2 + \lambda_i^2\right]}. \tag{24}$$

Assuming that the first instability encountered as P is increased is not a "static" one, the situation is as shown in Fig. 1. Curves (a) are the root loci of λ_i in the upper half of the

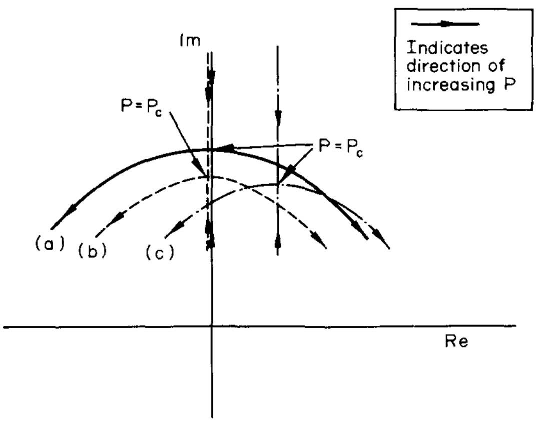

FIG. 1. The effect on the root loci of introducing a damping matrix proportional to the inertia matrix.

complex plane. The critical value, P_c, of P for oscillatory instability is that for coalescence of the roots. The effect of the term $(k(P)/2)^2$ under the radical in (24) is to modify the loci, but not to destroy the symmetry about the imaginary axis, as in curves (b). The effect of the term $-k(P)/2$ before the radical is to provide an overall shift into the right-hand or stable half-plane as shown by curves (c). At the critical value P_c of the undamped system, the corresponding root of the damped system has a negative real part $-k(P_c)/2$ from (24); therefore finite damping introduced in the form $\mathbf{b} = k(P)\mathbf{a}$ is always stabilizing.

The eigenvectors $\mathbf{\Phi}_i$ and $\mathbf{\Phi}_i^{**}$ satisfy respectively

$$[\mathbf{a}\lambda_i^2 + \mathbf{c}(P)]\mathbf{\Phi}_i = \mathbf{0} \tag{25}$$

$$[\mathbf{a}\lambda_i^{**2} + k(P)\mathbf{a}\lambda_i^{**} + \mathbf{c}(P)]\mathbf{\Phi}_i^{**} = \mathbf{0}. \tag{26}$$

But from (23), (26) can be written

$$[\mathbf{a}\lambda_i^2 + \mathbf{c}(P)]\mathbf{\Phi}_i^{**} = \mathbf{0} \tag{27}$$

so that $\mathbf{\Phi}_i^{**} = \mathbf{\Phi}_i$. The damped system eigenvectors are therefore the same as those of the undamped system which are real for $P \leq P_c$ and complex for $P > P_c$ provided at least one pair of eigenvalues remains complex.

An example of the stabilizing shift given to the eigenvalues is shown in Fig. 2. The system has four degrees of freedom and is taken from an aircraft wing aeroelastic flutter problem with the damping coefficients made proportional to the inertia coefficients. The numerical details appear in Ref. [11] with $k(P) = p_8/4p_7$. The chief parameter is airspeed on which the scalar $k(P)$ is linearly dependent for consistency with the nature of the naturally occurring aerodynamic damping. It is seen that the first instability occurring as the scaled airspeed V is increased is completely suppressed, and a second occurs at a higher value of V.

Simple expressions for the modal frequencies and rates of decay can be obtained in the binary case, and these may be found in Ref. [13].

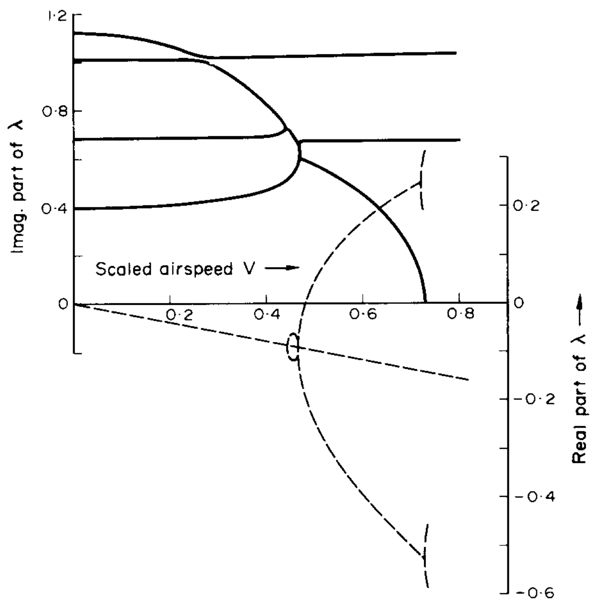

FIG. 2. Aircraft flutter roots for a case in which the damping matrix is proportional to the inertia matrix.

4. STABILIZING DAMPING FOR A TWO DEGREE OF FREEDOM SYSTEM

By concentrating attention on a system having only two degrees of freedom and of therefore limited complexity we can extend the search for stabilizing configurations of damping beyond that studied in the previous section.

The determinantal form of the characteristic equation may be written

$$\begin{vmatrix} \lambda^2 + b_{11}\lambda + c_{11} & b_{12}\lambda + c_{12} \\ b_{21}\lambda + c_{21} & \lambda^2 + b_{22}\lambda + c_{22} \end{vmatrix} = 0 \tag{28}$$

in which we let $\mathbf{a} = \mathbf{I}$ without loss of generality. The coefficients b_{ij} and c_{ij} may be functions of P.

This can be further reduced to

$$\begin{vmatrix} \Lambda^2+2\tau_{11}\Lambda+\Gamma_{11} & 2\tau_{12}\Lambda+\Gamma_{12} \\ 2\tau_{21}\Lambda+\Gamma_{21} & \Lambda^2+2\tau_{22}\Lambda+\Gamma_{22} \end{vmatrix} = 0 \tag{29}$$

where $\Lambda = \lambda/(c_{11}+c_{22})^{\frac{1}{2}}$, $\tau_{ij} = b_{ij}/(c_{11}+c_{22})^{\frac{1}{2}}$, and $\Gamma_{ij} = c_{ij}/(c_{11}+c_{22})^{\frac{1}{2}}$. A coefficient τ_{ij} is comparable to the "damping ratio" of a single degree of freedom system and $\Gamma_{11}+\Gamma_{22} = 1$. Expansion of the determinant into the polynomial form of the characteristic equation provides in the notation of (9) and (10)

$$p_4\lambda^4+\varepsilon\sigma_{31}\lambda^3+(p_2+\varepsilon^2\sigma_{22})\lambda^2+\varepsilon\sigma_{11}\lambda+p_0 = 0 \tag{30}$$

where

$$\begin{aligned} p_4 &= p_2 = 1 \\ p_0 &= \Gamma_{11}\Gamma_{22}-\Gamma_{12}\Gamma_{21} \\ \varepsilon\sigma_{31} &= 2(\tau_{11}+\tau_{22}) \\ \varepsilon^2\sigma_{22} &= 4(\tau_{11}\tau_{22}-\tau_{12}\tau_{21}) \\ \varepsilon\sigma_{11} &= 2(\tau_{11}\Gamma_{22}+\tau_{22}\Gamma_{11}-\tau_{12}\Gamma_{21}-\tau_{21}\Gamma_{12}). \end{aligned}$$

Flutter impends when Routh's third test function is zero

$$T_3^* = p_3^*p_2^*p_1^*-p_4^*p_1^{2*}-p_3^{*2}p_0^* = 0. \tag{31}$$

If damping is introduced in such a way that the critical value of the chief parameter, P_c, for incipient flutter remains unchanged then the equivalent of (31) for the undamped system

$$T_3 = p_4(p_2^2-4p_4p_0) = 0 \tag{32}$$

is also satisfied [from (13)].

Now, by scaling T_3^* appropriately it can be expressed in a form that contains the elements of T_3 [14], i.e.

$$T_3^* = \varepsilon^2\sigma_{31}^2\left\{\frac{T_3}{4p_4^2}+\frac{\varepsilon^2\sigma_{11}\sigma_{22}}{\sigma_{31}}-p_4\left(\frac{\sigma_{11}}{\sigma_{31}}-\frac{p_2}{2p_4}\right)^2\right\}. \tag{33}$$

Thus, for (31) and (32) to be satisfied simultaneously

$$\left\{\frac{\varepsilon^2\sigma_{11}\sigma_{22}}{\sigma_{31}}-p_4\left(\frac{\sigma_{11}}{\sigma_{31}}-\frac{p_2}{2p_4}\right)^2\right\}_{P=P_c} = 0 \tag{34}$$

and configurations of damping which when introduced into an initially undamped system do not alter the critical value of the chief parameter must satisfy this equation. Comprising a configuration are four independent damping parameters, τ_{11}, τ_{12}, τ_{21} and τ_{22}; in order to avoid undue complication and so that the regions of instability can be readily illustrated we make $\tau_{12} = \tau_{21}$. This would be the case anyway with damping that is purely structural in origin, although it would not necessarily apply in the aeroelastic flutter situation, for instance. The main independent parameters in (34) not concerned with damping are then $(\Gamma_{22}-\Gamma_{11})_{P=P_c} = A$ and $(\Gamma_{12}+\Gamma_{21})_{P=P_c} = B$, the first representing the modal frequency separation at $P = P_c$ and the second a measure of the coupling effect between the two degrees of freedom. It is assumed that divergence instability does not occur at a value of P lower than that for flutter, so that $\Gamma_{11}\Gamma_{22}-\Gamma_{12}\Gamma_{21} > 0$. It is also assumed that τ_{11} and τ_{22} are positive and that the dissipation function remains positive definite so that $\tau_{11}\tau_{22}-\tau_{12}^2 > 0$; by replacing τ_{12} by $\mu\sqrt{(\tau_{11}\tau_{22})}$ where $-1 < \mu < 1$ we automatically satisfy this condition.

Equation (34) may be re-written in terms of the newly defined parameters and it then takes the form

$$8(1-\mu^2)\tau_{11}\tau_{22}(\tau_{11}+\tau_{22})[(\tau_{11}+\tau_{22})+A(\tau_{11}-\tau_{22})-2B\mu\sqrt{(\tau_{11}\tau_{22})}] - [A(\tau_{11}-\tau_{22})-2B\mu\sqrt{(\tau_{11}\tau_{22})}]^2 = 0 \tag{35}$$

from which some useful observations may be made. The two terms always have opposing signs ($\varepsilon\sigma_{11}$ and hence the expression in square brackets is positive in the first term) and on looking back to (33) are seen to have opposing actions on the stability. Since $T_3^* > 0$ in the stable region, the first term in (35) is stabilizing and the second is destabilizing. The effect of putting the damping matrix proportional to the inertia matrix is to make $(\tau_{11}-\tau_{22}) = \tau_{12} = 0$ in (35). The second term vanishes and thus the configuration of damping is a stable one. However, this is only one configuration of many that cause the final term to disappear; by equating the term to zero it can be shown that any damping satisfying

$$\frac{\tau_{22}}{\tau_{11}} = \frac{(\mu^2+A^2/B^2)^{\frac{1}{2}}-\mu}{(\mu^2+A^2/B^2)^{\frac{1}{2}}+\mu} \tag{36}$$

is stabilizing. The combinations of values of τ_{11}, τ_{22} and μ which are either stabilizing or destabilizing when introduced into an undamped system at $P = P_c$ are found from (35) and shown in Fig. 3 for $A = B = 0{\cdot}4$. The wedge shaped region of stability that appears

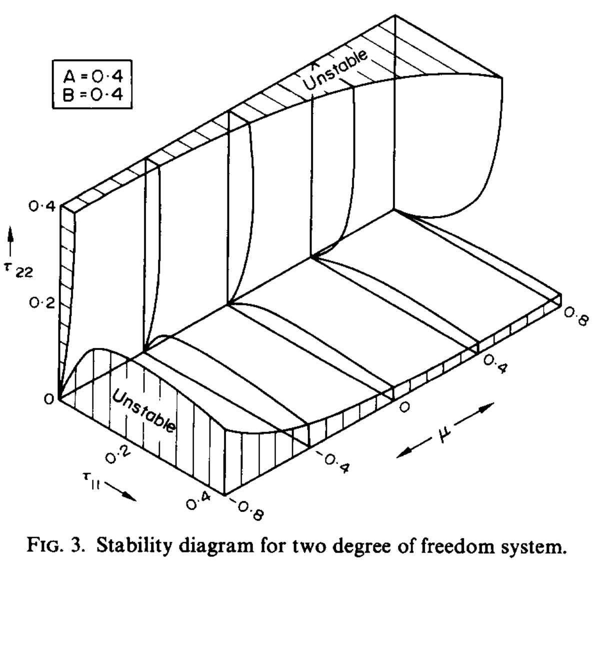

FIG. 3. Stability diagram for two degree of freedom system.

for small values of damping follows the slope defined by (36). The same picture may also be used to describe other combinations of the parameters, for if we let $f(\tau_{11}, \tau_{22}, A, B, \mu)$ be the function on the left-hand side of (35), it is readily seen that $f(\tau_{11}, \tau_{22}, A, B, \mu) = f(\tau_{22}, \tau_{11}, -A, B, \mu) = f(\tau_{11}, \tau_{22}, A, -B, -\mu)$.

The variations of the stability regions with A and B whilst μ is kept constant are shown in Figs. 4 and 5, respectively. The first case is interesting in that the slope of the "wedge" tends to zero as A becomes smaller but at $A = 0$, it disappears altogether. This is because the means of nulling the destabilizing term in (35) is removed and a completely unstable region exists for all small τ_{11} and τ_{22}. The case of $A = 0$ corresponds to a system for which the uncoupled modal frequencies coalesce at the undamped critical flutter speed. This occurs only when $\Gamma_{12}\Gamma_{21} = 0$ (see [11]) but the possibility of $B = \Gamma_{12}+\Gamma_{21} \neq 0$ still exists; one of the cross-dampings τ_{12} or τ_{21} combines with one of the cross-stiffnesses Γ_{21} or Γ_{12} to produce the destabilizing influence. The point along either axis at which the stability region begins is given by setting $A = 0$ in (35) and then letting $\tau_{ii} = 0$. This provides for

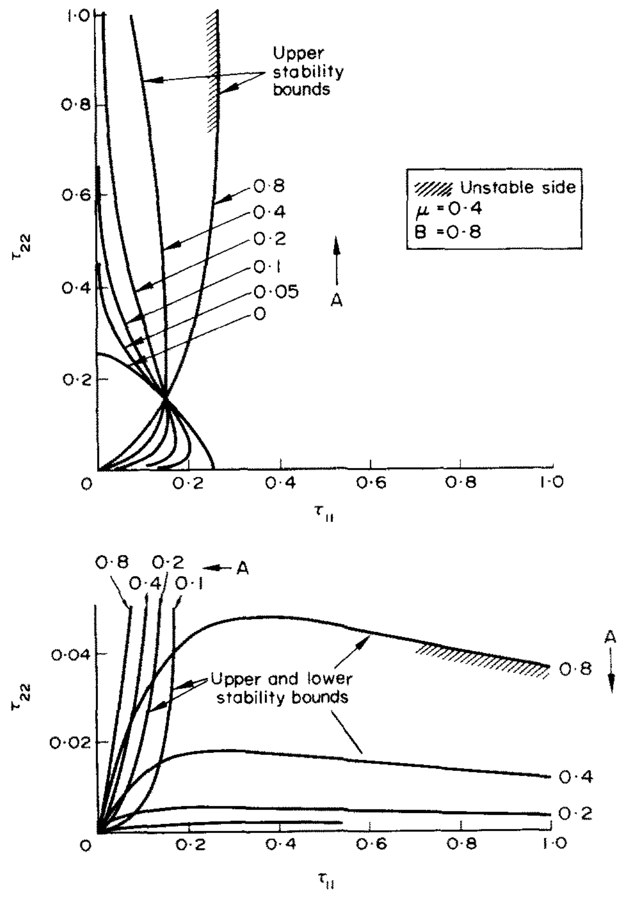

FIG. 4. Effect of varying A.

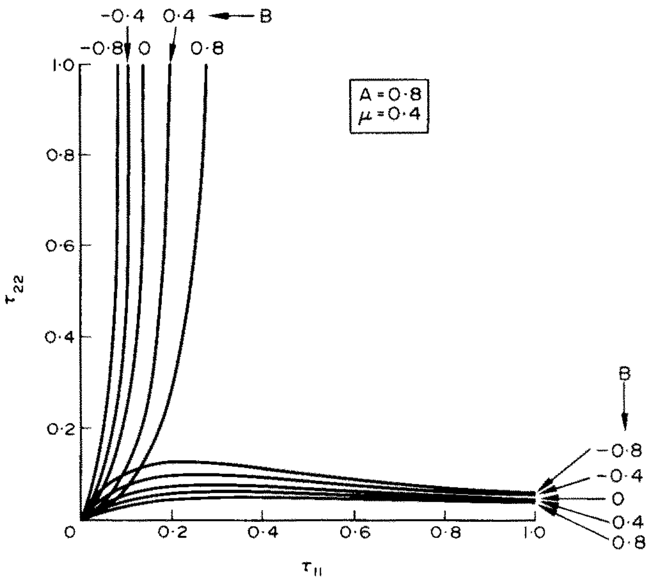

FIG. 5. Effect of varying B.

the other damping coefficient

$$\tau_{jj} = \frac{B\mu}{\sqrt{[2(1-\mu^2)]}}. \tag{37}$$

As the cross-damping coefficients are made smaller (i.e. μ is reduced) the region of instability also becomes less.

Figure 4 also shows a point common to the stability boundaries for varying A. This occurs when $\tau_{11} = \tau_{22}$ in (35) for then the equation is independent of A. Using the case $A = 0$ for simplicity it may be shown that the common point is given when

$$\tau_{11} = \tau_{22} = \frac{B\mu}{2\sqrt{[2(1-\mu^2)(1-B\mu)]}}. \tag{38}$$

In the example illustrated in Fig. 5 the regions of stability become thinner as B is increased and the wedge slope changes, but the basic shape characteristics remain. Curves similar to those in Figs. 4 and 5 were given in Salaün [15] for a system having no inertia or damping cross-coupling terms.

The generality of the damping configuration considered prevents explicit expressions for the modal frequencies and rates of decay from being obtained, except in the case of small damping which allows approximations to be made. Such expressions are found in Ref. [13] and graphical examples are given to show the various ways in which stability can be affected by damping.

The possible deviation of damping configurations away from the one given by (36) that maintain stability can be studied using (35) for the case of small damping. If we let the order of magnitude of a τ_{ij} be denoted by ε and that of its deviation be δ, then in (35) the first term is of order ε^4 and the second δ^2. Thus, the order of δ is ε^2, and a stabilizing damping configuration has damping coefficients $\tau_{ij}+\varepsilon\eta_{ij}$ where η_{ij} and τ_{ij} are both of order ε. The wedge-shaped stability region seen in Figs. 3–5 therefore virtually collapses onto the line defined by (36) for very small values of damping.

5. DISCUSSION

The introduction into an initially undamped nonconservative system of small damping is usually destabilizing. However, it has been shown that certain configurations of damping or damping matrix have a stabilizing influence. The case in which the damping matrix is proportional to the inertia matrix is one such example, and is relatively amenable to analysis. It is seen that the effect of the damping on the root loci is to impose an overall shift into the stable or right-hand half of the complex plane, which results in the system having a stable root where previously it had an incipiently unstable root. The eigenvectors at the same values of the chief parameter are left unchanged, which means that at the new point of instability the eigenvector associated with the flutter root is complex. When the damping is made very small the system tends towards the undamped system in its most important aspect, that of the critical value of the chief parameter for oscillatory instability. This is demonstrated by the appropriate Routh's test function tending to that of the undamped system.

In the two degree of freedom case the algebra is sufficiently simple to allow other damping configurations to be considered, in particular one in which the cross-dampings are equal. For this, it is seen that for damping ratios of the order of 10 or 20 per cent there is considerable scope for choosing a set of values that is stabilizing. However, for very small values, say of the order of 1 per cent, there would be virtually only one value of damping ratio given the other two; in fact, the variation on this value to maintain stability would again be about 1 per cent.

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Абстракт—Исследуется частная форма демифирования, которая всегда имеет стабилизационное влияние для неконсервативной системы. Указано, что для очень малого демифирования, критическое значение главного параметра для запождающегося флаттера сводится к такому же для незатухающей системы.

Для самого более простого слчая для хвух степеней свободы, исследуется диапазон форм демифирования, которые обеспечивают устойчивость.